

RADII OF REGULAR POLYTOPES

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ABSTRACT. This paper deals with the three types of regular polytopes which exist in all dimensions – regular simplices, cubes and regular cross-polytopes – and their outer and inner radii. While the inner radii of regular simplices are well studied, only a few cases are solved for the outer radii. We give a lower bound on these radii, and show that this bound is tight in almost 3 out of 4 dimensions. In a further section we complete the results about inner and outer radii of general boxes and cross-polytopes. Finally, because cubes and regular cross-polytopes are radii-minimal projections of simplices, we show that it is possible to deduce the results about their radii from the results about the outer radii of simplices.

1. INTRODUCTION AND BASIC NOTATIONS

There are three classes of regular polytopes which last in general d -space: regular simplices, (hyper-) cubes, and regular cross-polytopes. In this paper we investigate the inner and outer j -radii of this polytopes. Here the inner j -radius of a body is defined as the radius of a biggest j -ball fitting into the body, and the outer j -radius as the smallest radius of a circumball of an orthogonal projection of the body onto any j -space.

While the inner radii of regular simplices are well studied [1], very less is known about their outer radii. We give a lower bound on these radii, and show that this bound is tight in almost 3 out of 4 cases. An important step towards this result is the investigation of quasi isotropic polytopes (Kawashima called them π -polytopes [9], but we prefer to call them isotropic as they are in an isotropic position in the sense of [5]). Specifically, we will show that the existence of a quasi-isotropic j -dimensional polytope with $d + 1$ vertices is equivalent to the existence of a projection of the regular simplex such that the lower bound is attained.

In a further section we investigate the radii of general boxes and cross-polytopes. While the inner radii of boxes were computed in [4], nothing could be found about their outer radii in the literature. We close this gap, as well as we transfer the

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results about boxes to results about cross-polytopes via polarization. Finally, we show that the radii of cubes and regular cross-polytopes can be obtained almost completely from our results about the outer-radii of regular simplices.

Let $\mathbb{E}^d = (\mathbb{R}^d, \|\cdot\|)$ denote the d -dimensional Euclidean space, $d \geq 2$, \mathbb{B} and \mathbb{S} the unit ball and the unit sphere in \mathbb{E}^d , and $\langle \cdot, \cdot \rangle$ the usual *scalar product* $\langle x, y \rangle = x^T y$. Furthermore, we use $\{e_1, \dots, e_d\}$ for the *standard basis* of \mathbb{E}^d .

A set $C \subset \mathbb{E}^d$ is called a *body* if it is bounded, closed, convex and contains an inner point. For every body $C \subset \mathbb{E}^d$ let $C^\circ = \{y \in \mathbb{R} : \langle c, y \rangle \leq 1 \text{ for all } c \in C\}$ denote the *polar* of C .

By $\mathcal{L}_{j,d}$ and $\mathcal{A}_{j,d}$ we denote the set of all j -dimensional linear subspaces and all j -dimensional affine subspaces of \mathbb{E}^d , respectively. For any $F \in \mathcal{L}_{j,d}$ let $F^\perp \in \mathcal{L}_{d-j,d}$ be the orthogonal space of F . Let $\text{span}\{s_1, \dots, s_j\}$ denote the *linear span* $\{x \in \mathbb{R}^d : x = \sum_{k=1}^j \lambda_k s_k, \lambda_k \in \mathbb{R}\}$ of $s_1, \dots, s_j \in \mathbb{S}$. For any set $A \in \mathbb{E}^d$, $A|F$ denotes the (*orthogonal*) *projection* of A onto $F \in \mathcal{L}_{j,d}$. If s_1, \dots, s_j is an orthonormal basis of F we also use A_{s_1, \dots, s_j} instead of $A|F$ and A^{s_1, \dots, s_j} for the projection of A onto F^\perp . For any $x \in \mathbb{E}^{d_1}$ and $y \in \mathbb{E}^{d_2}$ let $x \otimes y$ denote the matrix with elements $x_i y_j$, $i = 1, \dots, d_1$ and $j = 1, \dots, d_2$ and note that for any set of orthonormal vectors $\{s_1, \dots, s_j\}$ the (orthogonal) projection P of \mathbb{E}^d onto $\text{span}\{s_1, \dots, s_j\}$ can be represented by the matrix $\sum_{l=1}^j s_l \otimes s_l$.

For any two sets $A, B \subseteq \mathbb{E}^d$ the *Minkowski sum* $A + B$ is defined as $A + B = \{a + b \in \mathbb{E}^d : a \in A, b \in B\}$. Now for any $j \in \{1, \dots, d\}$ the *inner j -radius* $r_j(C)$ of a convex set C is defined by

$$r_j(C) = \max \{\rho \geq 0 : (q + \rho \mathbb{B}) \cap F \subset C, q \in F \in \mathcal{A}_{j,d}\}$$

and the *outer j -radius* $R_j(C)$ by

$$R_j(C) = \min \{\rho \geq 0 : E + \rho \mathbb{B} \supset C, E \in \mathcal{A}_{d-j,d}\}.$$

If a body C_1 arises from C_2 by rotation, translation and dilatation, we say C_1 is *similar* to C_2 . Note that the radii of a body do not change if the body is translated or rotated; neither are the relationships of the radii affected by scaling the body. For this reason, we will often use the word ‘ball’ to signify any similar copy of \mathbb{B} , and the same we do for simplices, cross-polytopes and cubes.

Let T^d denote the regular d -simplex $\text{conv}\{e_1, \dots, e_{d+1}\}$ (embedded in \mathbb{E}^{d+1}) and X^d the cross-polytope $\text{conv}\{\pm e_1, \dots, \pm e_d\}$, where e_k denotes the k -th unit vector (of the appropriate space). By B_{a_1, \dots, a_d} we denote a d -dimensional box of the form $\{x \in \mathbb{R}^d : -a_i \leq x_i \leq a_i, i \in \{1, \dots, d\}\}$ and the cube $B_{1, \dots, 1}$ we denote by B^d .

2. REGULAR SIMPLICES

The following result about the inradii of simplices is taken from [1]:

Proposition 2.1. *For the inner radii of the regular simplex of edge length $\sqrt{2}$ it holds $r_j(T^d) = \sqrt{\frac{1}{j(j+1)}}$.*

So we can concentrate on the outer-radii. A proof of the following proposition can be found in [6]:

Proposition 2.2. *If C is a symmetric body and $1 \leq j \leq d$ then $r_j(C)R_j(C^0) = 1$ and $R_j(C)r_j(C^0) = 1$.*

Now we state the so far known results about outer radii of regular simplices, which are taken from [8], [10], and [11] respectively.

Proposition 2.3.

$$\begin{aligned} (i) \quad R_d(T^d) &= \sqrt{\frac{d}{d+1}} \\ (ii) \quad R_1(T^d) &= \begin{cases} \sqrt{\frac{1}{d+1}}, & \text{if } d \text{ odd} \\ \sqrt{\frac{d+1}{d(d+2)}}, & \text{if } d \text{ even.} \end{cases} \\ (iii) \quad R_{d-1}(T^d) &= \sqrt{\frac{d-1}{d+1}}, \text{ if } d \text{ is odd.}^1 \end{aligned}$$

Proposition 2.3 is not as complete a result as Proposition 2.1. At the end of this section we will be able to give a result on the outer radii of regular simplices which is much more general than the above Proposition. To do so, we make use of the following definition:

Definition 2.4. *We call any set of orthonormal vectors $\{s_1, \dots, s_j\}$, $j \in \{1, \dots, d\}$ in \mathbb{E}^{d+1}*

- (i) *a valid subset basis (vsb for short) if $\sum_{k=1}^{d+1} s_{lk} = 0$ for all $l \in \{1, \dots, j\}$, and*
- (ii) *a good subset basis (gsb for short) if it is a vsb and $\sum_{l=1}^j s_{lk}^2 = \frac{j}{d+1}$ for all $k \in \{1, \dots, d+1\}$.*

Note that any set of orthonormal vectors $\{s_1, \dots, s_j\}$ is called a vsb if it spans a j -dimensional subspace of $\mathbb{E}_0^{d+1} = \left\{x \in \mathbb{E}^{d+1} : \sum_{k=1}^{d+1} x_k = 0\right\}$, the d -dimensional linear subspace of \mathbb{E}^{d+1} parallel to the hyperplane in which we have embedded T^d .

The projection of T^d onto \mathbb{E}_0^{d+1} can be written as $I^{d+1} - \frac{1}{d+1}\mathbf{1}^{d+1}$, where I^{d+1} denotes the identity matrix in $\mathbb{E}^{(d+1) \times (d+1)}$ and $\mathbf{1}^{d+1}$ the matrix in $\mathbb{E}^{(d+1) \times (d+1)}$ consisting only of 1's. Hence it holds that $\sum_{l=1}^d s_l \otimes s_l = I^{d+1} - \frac{1}{d+1}\mathbf{1}^{d+1}$, for every vsb of d elements. This enables us to obtain the important fact that each vsb is a gsb if $j = d$, which we use in Corollary 2.6.

Now we start improving the results on the outer radii of regular simplices by giving a general lower bound, which we will prove to be tight in many cases further on. This theorem will also show the reason why we call a vsb good if it fulfills the condition (ii) in Definition 2.4.

¹Note that the proof of the even-case result provided in [12] is wrong (see [2]), but it is resettled in recent work [3]

Theorem 2.5. $R_j(T^d) \geq \sqrt{\frac{j}{d+1}}$ for all $j \in \{1, \dots, d\}$ and equality holds iff there exists a gsb $\{s_1, \dots, s_j\}$ in \mathbb{E}^{d+1} .

Proof. Let P denote the projection onto some subspace spanned by a vsb $\{s_1, \dots, s_j\}$. It follows

$$\|Pe_k\|^2 = \langle Pe_k, e_k \rangle = \left\langle \sum_{l=1}^j s_{lk}s_l, e_k \right\rangle = \sum_{l=1}^j s_{lk}^2.$$

Now assume there exists any $x \in \mathbb{E}^{d+1}$ such that $\|x - Pe_k\|^2 < \sqrt{\frac{j}{d+1}}$ for all $k = 1, \dots, d+1$. Summing over the k 's it follows

$$\begin{aligned} j &> \sum_{k=1}^{d+1} \|x - Pe_k\|^2 \\ &= \sum_{k=1}^{d+1} (\|x\|^2 - 2\langle x, Pe_k \rangle + \|Pe_k\|^2) \\ &= (d+1)\|x\|^2 - 2 \left\langle x, \sum_{k=1}^{d+1} \sum_{l=1}^j s_{lk}s_l \right\rangle + \sum_{k=1}^{d+1} \sum_{l=1}^j s_{lk}^2 \end{aligned}$$

and since $\sum_{k=1}^{d+1} s_{lk} = 0$ and $\sum_{k=1}^{d+1} s_{lk}^2 = 1$

$$\begin{aligned} &= (d+1)\|x\|^2 + j \\ &\geq j \end{aligned}$$

which is a contradiction. This proves the first part of the theorem. To prove the other part, look at the expression above; it is easy to see that equality in $\|x - Pe_k\|^2 \leq \frac{j}{d+1}$ for all k can only be obtained if $x = 0$ and $\sum_{l=1}^j s_{lk}^2 = \frac{j}{d+1}$. \square

As every vsb of d vectors is already a gsb we receive the following corollary from Theorem 2.5 from the basis extension property (used on \mathbb{E}_0^{d+1}):

Corollary 2.6. For any dimension d and any $j \in \{1, \dots, d-1\}$ it holds $R_j(T^d) = \sqrt{\frac{j}{d+1}}$ iff $R_{d-j}(T^d) = \sqrt{\frac{d-j}{d+1}}$ holds. Moreover the optimal projections take place in orthogonal subspaces.

Corollary 2.6 shows that Proposition 2.3 (ii) and (iii) correspond to each other in the sense that the lower bound of Theorem 2.5 is attained in both cases for odd dimensions and that the bound is not attained in even dimensions.

The following Proposition is a polar-version of a theorem due to John [7]:

Proposition 2.7. \mathbb{B} is the ellipsoid of minimal volume containing some body $C \subset \mathbb{E}^d$ iff $C \subset \mathbb{B}$ and for some $m \geq d$ there are unit vectors u_1, \dots, u_m on the boundary of C , and positive numbers c_1, \dots, c_m summing to d such that

$$(i) \sum_{i=1}^m c_i u_i = 0, \text{ and}$$

$$(ii) \sum_{i=1}^m c_i u_i \otimes u_i = I^d.$$

It is obvious that if C is a regular polytope all c_i can be chosen as $\frac{d}{m}$ where m is the number of vertices of C . But it is not obvious which other polytopes fulfill this property. Nevertheless, according to [5] these polytopes are in an isotropic position, corresponding to the measure μ^* on \mathbb{S} that gives mass $\frac{d}{m}$ to all vertices u_i . This is the source for the following definition:

Definition 2.8. Let $C = \text{conv}\{u_1, \dots, u_m\} \subset \mathbb{B}$ be a polytope, where all u_i 's are situated on \mathbb{S} . We call C quasi isotropic, if all the c_i 's in Proposition 2.7 can be taken as $\frac{d}{m}$, and isotropic, if additionally $u_{i_1} \neq u_{i_2}$ for all $i_1 \neq i_2$.

Lemma 2.9. There exists a gsb s_1, \dots, s_j of \mathbb{E}^{d+1} iff there exists a quasi-isotropic polytope $C = \text{conv}\{u_1, \dots, u_{d+1}\} \subset \mathbb{E}^j$, $j \leq d$. Moreover if we project T^d onto $\text{span}\{s_1, \dots, s_j\}$ the projection will be similar to the corresponding C .

Proof. If $C = \text{conv}\{u_1, \dots, u_{d+1}\}$ is a quasi isotropic polytope then

- (i) $\|u_k\| = 1$,
- (ii) $\sum_{k=1}^{d+1} u_k = 0$, and
- (iii) $\sum_{k=1}^{d+1} u_k \otimes u_k = \frac{d+1}{j} I^j$.

Now let $s_l = \sqrt{\frac{j}{d+1}}(u_{1,l}, \dots, u_{d+1,l})^T$, $l = 1, \dots, j$. This defines a gsb. For showing this it is necessary that the s_l form an orthonormal set, but this is the case because of (iii). $\sum_{k=1}^{d+1} s_{lk}$ has to be 0, but this follows from (ii), and finally we need $\sum_{l=1}^j s_{lk}^2 = \frac{j}{d+1}$ for all k , but this is true because of (i). The other direction can be shown using a similar reasoning.

Now, if we project the vertices of T^d onto $\text{span}\{s_1, \dots, s_j\}$ we get $Pe_k = \sum_{l=1}^j s_{lk} s_l = \sum_{l=1}^j \sqrt{\frac{j}{d+1}} u_{kl} s_l$. Hence the values $\sqrt{\frac{j}{d+1}} u_{kl}$ are just the coordinates of the vertices of the projection in terms of the basis s_1, \dots, s_j . \square

Lemma 2.9 can be used in 2 ways:

- (i) We know that $R_j(T^d) = \sqrt{\frac{j}{d+1}}$ whenever we find a quasi-isotropic j -dimensional polytope with $d+1$ vertices and vice versa (therefore, due to Proposition 2.3 (iii) there cannot be quasi-isotropic polytopes with $d+2$ vertices if d is odd), and
- (ii) we know that $R_k(C) = \sqrt{\frac{k}{j}}$ for any $k \leq j$ such that the gsb $\{s_1, \dots, s_j\}$ can be split into 2 gsb's $\{s_1, \dots, s_k\}$ and $\{s_{k+1}, \dots, s_j\}$.

We will first concentrate our attention to (i) but come back to (ii) later. Because every m -gon is a regular body in \mathbb{E}^2 , and because a prism or an anti-prism of such an m -gon, such that all the vertices are on the unit sphere, is at least isotropic, we receive (always keeping Corollary 2.6 in mind) that:

Corollary 2.10. (i) $R_2(T^d) = \sqrt{\frac{2}{d+1}}$, $R_{d-2}(T^d) = \sqrt{\frac{d-2}{d+1}}$ for all $d \geq 2$, and
(ii) $R_3(T^d) = \sqrt{\frac{3}{d+1}}$, $R_{d-3}(T^d) = \sqrt{\frac{3}{d+1}}$ for all odd $d \geq 3$.

In the following we will not mention the $(d-j)$ -cases as long as we make no special use of them.

Lemma 2.11. (i) Suppose $d = d_1 + d_2 + 1$ and $R_j(T^{d_1}) = \sqrt{\frac{j}{d_1+1}}$ and $R_j(T^{d_2}) = \sqrt{\frac{j}{d_2+1}}$. Then it also holds $R_j(T^d) = \sqrt{\frac{j}{d+1}}$.
(ii) Suppose $d + 1 = (d_1 + 1)(d_2 + 1)$ and $R_{j_1}(T^{d_1}) = \sqrt{\frac{j_1}{d_1+1}}$ and $R_{j_2}(T^{d_2}) = \sqrt{\frac{j_2}{d_2+1}}$. Then $R_j(T^d) = \sqrt{\frac{j}{d+1}}$ for all $j \in \{k_1, k_2, k_1 k_2, k_1(k_2 + 1), (k_1 + 1)k_2, (k_1 + 1)(k_2 + 1)\}$, where $k_i \in \{j_i, d_i - j_i, d_i\}$, $i = 1, 2$.

Proof. Part (i) is quite simple in terms of the polytopes: If C_1 and C_2 are two quasi isotropic polytopes with $d_1 + 1$ and $d_2 + 1$ vertices, respectively, then their convex hull has $d + 1$ vertices and is again quasi isotropic. For part (ii) there is a bit more to do. Suppose $\{s_1, \dots, s_{k_1}\}$ and $\{t_1, \dots, t_{k_2}\}$ are gsb's in \mathbb{E}^{d_1+1} and \mathbb{E}^{d_2+1} , respectively, and consider the following three sets in \mathbb{E}^{d+1} :

$$\sqrt{\frac{1}{d_2+1}} \underbrace{(s_l^T, \dots, s_l^T)^T}_{d_2+1}, \quad l = 1, \dots, k_1,$$

and

$$\sqrt{\frac{1}{d_1+1}} \underbrace{(t_{l1}, \dots, t_{l1})}_{d_1+1}, \dots, \underbrace{(t_{l(d_2+1)}, \dots, t_{l(d_2+1)})}_{d_1+1}^T, \quad l = 1, \dots, k_2,$$

and

$$s_{l_1} \otimes t_{l_2}, \quad l_1 = 1, \dots, k_1, \quad l_2 = 1, \dots, k_2$$

where we take the \otimes -matrix as a vector, column by column. It is easy to see that all vectors in the three sets form a vsb of size $k_1 + k_2 + k_1 k_2$. Now we only need that each of the three sets forms a gsb. But again this is obvious for the first two sets and not hard to see for the last one. \square

One should note that the first two groups could also be obtained by Part (i) applying it k_i -times, and that the polytope corresponding to the $k_1 k_2$ -gsb is a homothetic of $\text{conv}\{u_{i_1} \otimes v_{i_2}, \quad i_1 = 1, \dots, d_1 + 1, \quad i_2 = 1, \dots, d_2 + 1\}$ if $\text{conv}\{u_1, \dots, u_{d_1+1}\}$ and $\text{conv}\{v_1, \dots, v_{d_2+1}\}$ where the polytopes corresponding to the initial gsb's.

The polytopes one gets out of Part (ii) stay to be much more 'regular' as the ones obtained Part (i).

Example 2.12. Suppose $d_1 = 1$ and $d_2 = 2$ in Lemma 2.11, so $d = 5$. If we forget about the normalizing factors gsb's for d_1 and d_2 could be

$$\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\},$$

respectively. Using the construction in Lemma 2.11 we get the following three gsb's for $d = 5$:

(i) the d_1 -gsb by putting the $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ vector $d_2 + 1 = 3$ times below each other

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\},$$

(ii) the d_2 -gsb by taking every entry in the original d_2 -gsb $d_1 + 1$ times below each other

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -2 \\ -2 \end{pmatrix} \right\},$$

and

(iii) the $d_1 d_2$ -gsb by multiplying each vector of the d_1 -gsb coordinate wise with any vector of the d_2 -gsb

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ -2 \\ 2 \end{pmatrix} \right\}.$$

Now, we derive the main theorem by making use of Corollary 2.6 and Lemma 2.11(i):

Theorem 2.13. $R_j(T^d) = \sqrt{\frac{j}{d+1}}$, if

(i) d is odd, or

(ii) j is even and $d \neq 2j$.

Proof. We do an inductive proof over j and d . We know already that (i) and (ii) are true for $j = 1, 2, 3$. So let $j \geq 4$. Now suppose $d < 2j$. Then $d - j < j$ and therefore the statement follows inductively by applying Corollary 2.6, because if

d is odd then we do not depend on j and if d is even then $d - j$ is even if j is and $2(d - j) = d$ would mean $2j = d$. If $d > 2j$ then we can apply Lemma 2.11 (i) with $R_j(T^j)$ and $R_j(T^{d-j-1})$ or in the case that $d - j = 2j$ with $R_j(T^{j+2})$ and $R_j(T^{d-j-3})$. $R_j(T^j)$ and $R_j(T^{j+2})$ belong to the $(d < 2j)$ -case and the other two we can use by induction if $d - j - 1$ or $d - j - 3$ are good for one of the two cases. But if j is odd we can assume that d is odd and then this two numbers are also odd and we fulfill case (i). On the other hand, if j is even at least one of them is not equal to $2j$ and we have case (ii) for at least one of the two pairs. We are left with the case $d = 2j$. But this can only be in case of even d and hence we aren't in (i) or (ii). \square

The 'only if'-direction in Theorem 2.13 would not be true as one can find gsb's for the special case that $d + 1 = 2j$, with even j for many d :

Lemma 2.14. *In case of $d = 2j$ and j even $R_j(T^d) = \sqrt{\frac{j}{d+1}}$, if $d + 1 = (d_1 + 1)(d_2 + 1)$ with d_1 divides j and if $\frac{j}{d_1}$ is odd then $\frac{j}{d_1} - 1 \neq \frac{d_1}{2}$.*

Proof. Because d_1 divides j we can make use of Lemma 2.11 (ii) with $j_1 = d_1$ and $j_2 = \frac{j}{d_1}$ or $j_2 = \frac{j}{d_1} - 1$ which ever is even. Now we only have to make sure that $2j_2 \neq d_2$. But from $2\frac{j}{d_1} = d_2$ follows $d + 1 = (d_1 + 1)(\frac{2j}{d_1} + 1) = d + \frac{2j}{d_1} + d_1 + 1$ and therefore $j = -\frac{d_1^2}{2}$ which is a contradiction. And from $2\frac{j}{d_1} - 1 = d_2$ follows $d + 1 = (d_1 + 1)(\frac{2j}{d_1} - 1) = d + \frac{2j}{d_1} - d_1 - 1$ and therefore $\frac{j}{d_1} = \frac{d_1}{2} + 1$ the case excluded by the assumption. \square

Lemma 2.14 includes the case that 3 divides $d + 1$ (because j is even) and the case that $d + 1 = (d_1 + 1)^2$ with 4 does not divide d_1 (because $j = \frac{d_1}{2} + 1$).

On the other hand Lemma 2.11 (ii) cannot help in the case $d = 2j$, j even if $d + 1$ is prime neither does it help always if $d + 1$ is not prime since if $d + 1 = 5 \cdot 17$ we have $j = 42$. Here we would need $j_1 \in \{2, 4\}$, but if $j_1 = 2$ it follows $j_2 > 16$ and $j_1 = 4$ is not possible because neither 4 nor 5 divides 42.

3. BOXES AND CROSS-POLYTOPES

Proposition 3.1. *Let $0 < a_1 \leq \dots \leq a_d$. Then*

(i)

$$r_j(B_{a_1, \dots, a_d}) = \sqrt{\frac{a_1^2 + \dots + a_{d-k}^2}{j - k}},$$

where k is the smallest of the integers $0, \dots, j - 1$ that satisfies

$$a_{d-k} \leq \sqrt{\frac{a_1^2 + \dots + a_{d-k-1}^2}{j - k - 1}},$$

and

j, d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-
2		+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
3			+	-	+	(-)	+	(-)	+	(-)	+	(-)	+	(-)	+	(-)
4				+	+	+	+	+	+	+	+	+	+	+	+	+
5					+	-	+	(-)	+	(-)	+	(-)	+	(-)	+	(-)
6						+	+	+	+	+	+	?	+	+	+	+
7							+	-	+	(-)	+	(-)	+	(-)	+	(-)
8								+	+	+	+	+	+	+	+	?
9									+	-	+	(-)	+	(-)	+	(-)
10										+	+	+	+	+	+	+

TABLE 1. The table shows the existence of j -dimensional quasi isotropic polytopes with $d + 1$ vertices. The first column states the j value, the first row the value of d . A ‘+’ indicates the existence, a ‘-’ the non-existence. The ‘(-)’ entries show that the nonexistence is not proven but very unlikely, the ‘?’-s show the open cases for even j . Be careful, in terms of the outer j -radii both ‘+’ and ‘-’, indicate that the radii of the regular simplices are known, and each ‘(-)’ or ‘?’ entry stands for an unsolved case.

(ii)

$$R_j(X_{a_1, \dots, a_d}) = \sqrt{\frac{(j-k) \prod_{i=k}^d a_i^2}{\sum_{i=k}^d \prod_{l \neq i} a_l^2}},$$

where k is the smallest of the integers $0, \dots, j-1$ that satisfies

$$a_k \geq \sqrt{\frac{(j-k-1) \prod_{i=k+1}^d a_i^2}{\sum_{i=k+1}^d \prod_{l \neq i} a_l^2}}.$$

The corresponding result about the outer radii of boxes seems to be very intuitive. It says that one should just project the box through one of its smallest faces. One gets the inner radii of cross-polytopes from polarization. Before we state the final theorem, we give a technical lemma which will be useful in the proof of the theorem.

Lemma 3.2. *Let $s_l \in \mathbb{E}^d$, $l = 1, \dots, j$, $j \leq d$, $d \geq 2$ be a set of orthonormal vectors and $a_1, \dots, a_d \in \mathbb{R}_+$. Then there exists a choice of plus and minus signs in $\sum_{l=1}^j (\sum_{k=1}^d \pm a_k s_{lk})^2$ such that this is at least $\sum_{k=1}^d a_k^2 \sum_{l=1}^j s_{lk}^2$.*

Proof. Essentially we have to show that there is an $\alpha = (\alpha_1, \dots, \alpha_d)$ with $\alpha_k \in \{-a_k, a_k\}$ such that

$$\Gamma_\alpha := \sum_{1 \leq k_1 < k_2 \leq d} \alpha_{k_1} \alpha_{k_2} \zeta_{k_1, k_2} \geq 0, \text{ where } \zeta_{k_1, k_2} := \sum_{l=1}^j s_{lk_1} s_{lk_2}.$$

This is done by an inductive proof.

First consider the case $d = 2$. Then $\Gamma_\alpha = \alpha_1 \alpha_2 \zeta_{1,2}$. So we choose $\alpha_i = a_i$, $i = 1, 2$ if $\zeta_{1,2} \geq 0$ and if $\zeta_{1,2} < 0$ we choose $\alpha_1 = a_1$ and $\alpha_2 = -a_2$.

Now let $d = 3$. Hence $\Gamma_\alpha = \alpha_1 \alpha_2 \zeta_{1,2} + \alpha_1 \alpha_3 \zeta_{1,3} + \alpha_2 \alpha_3 \zeta_{2,3}$. Now suppose $\Gamma_{(a_1, a_2, -a_3)} < 0$ and $\Gamma_{(-a_1, a_2, -a_3)} < 0$. It follows that

$$0 > \Gamma_{(a_1, a_2, -a_3)} + \Gamma_{(-a_1, a_2, -a_3)} = -2a_2 a_3 \zeta_{2,3}$$

and therefore that $\zeta_{2,3} > 0$. Analogously one can show that $\zeta_{1,2}$ and $\zeta_{1,3}$ are positive; but then we can choose $\alpha = (a_1, a_2, a_3)$.

By knowing that the statement is correct for $d = 2, 3$ we can take an inductive step of 2, that means we assume the statement is proven up to some d and now conclude that it is also true for $d + 2$.

Now suppose the statement would be wrong for $d + 2$, meaning $\Gamma_\alpha < 0$ for all possible choices of $\alpha \in \mathbb{E}^{d+2}$. Hence

$$\begin{aligned} 0 &> \Gamma_{\alpha_1, \dots, \alpha_d, a_{d+1}, a_{d+2}} + \Gamma_{\alpha_1, \dots, \alpha_d, -a_{d+1}, a_{d+2}} + \Gamma_{\alpha_1, \dots, \alpha_d, a_{d+1}, -a_{d+2}} + \Gamma_{\alpha_1, \dots, \alpha_d, -a_{d+1}, -a_{d+2}} \\ &= 4 \sum_{1 \leq k_1 < k_2 \leq d} \alpha_{k_1} \alpha_{k_2} \zeta_{k_1, k_2}. \end{aligned}$$

However, this is not possible as by the induction hypothesis

$$\sum_{1 \leq k_1 < k_2 \leq d} \alpha_{k_1} \alpha_{k_2} \zeta_{k_1, k_2} \geq 0$$

for at least one possible choice of α . □

Theorem 3.3. *Let $0 < a_1 \leq \dots \leq a_d$. Then*

$$(i) \ R_j(B_{a_1, \dots, a_d}) = \sqrt{a_1^2 + \dots + a_j^2}, \text{ and}$$

$$(ii) \ r_j(X_{a_1, \dots, a_d}) = \frac{\prod_{i=d-j+1}^d a_i}{\sqrt{\sum_{i=d-j+1}^d \prod_{l \neq i} a_l^2}}.$$

Proof. It suffices to show Part (i), Part (ii) follows then from Proposition 2.2, and as the result is obvious if $d = 1$ we can assume that $d \geq 2$. Any vertex v of B_{a_1, \dots, a_d} can be written in the form $v = \sum_{k=1}^d \pm a_k e_k$ and all possible choices of the plus and minuses in that formula lead to a vertex of B_{a_1, \dots, a_d} . Hence, for every projection $P = \sum_{l=1}^j s_l \otimes s_l$ with pairwise orthogonal unit-vectors $s_l \in \mathbb{E}^d$, it holds that $\|Pv\|^2 = \sum_{l=1}^j \langle v, s_l \rangle^2 = \sum_{l=1}^j (\sum_{k=1}^d \pm a_k s_{lk})^2$ and because of Lemma 3.2 there exists a vertex of B_{a_1, \dots, a_d} such that this is at least $\sum_{k=1}^d a_k^2 \sum_{l=1}^j s_{lk}^2$. Now extend the set $\{s_1, \dots, s_j\}$ to an orthonormal basis of \mathbb{E}^d . As $\sum_{l=1}^d s_l \otimes s_l = I$

it follows that $\sum_{k=1}^d s_{lk}^2 = \sum_{l=1}^d s_{lk}^2 = 1$, for all $k = 1, \dots, d$, and therefore that $t_k := \sum_{l=1}^j s_{lk}^2 \in [0, 1]$. Now, because $\sum_{k=1}^d t_k = \sum_{l=1}^j \sum_{k=1}^d s_{lk}^2$ has to equal j the minimum value of $\sum_{k=1}^d t_k a_k^2$ will be achieved for $t_1 = \dots = t_j = 1$ and $t_{j+1} = \dots = t_d = 0$. Hence $R_j(B_{a_1, \dots, a_d}) \geq \sqrt{a_1^2 + \dots + a_j^2}$. But as the projection of B_{a_1, \dots, a_d} through its j -face B_{a_1, \dots, a_j} achieves this value we get the desired result. \square

Of course, one can easily get the radii of cubes and regular cross-polytopes from the results about the radii of general boxes and cross-polytopes:

Corollary 3.4. *The following hold:*

- (i) $r_j(B^d) = \sqrt{\frac{d}{j}}$, and
- (ii) $R_j(X^d) = \sqrt{\frac{j}{d}}$.
- (iii) $R_j(B^d) = \sqrt{j}$, and
- (iv) $r_j(X^d) = \sqrt{\frac{1}{j}}$.

Proof. Part (i) and (ii) follow from Proposition 3.1 by choosing $k = 0$ there, Part (iii) and (iv) from Theorem 3.3. \square

However, one should recognize that we can prove Corollary 3.4 almost without using Proposition 3.1 and 3.3. Except in the case where $2j = d - 1$ and Lemma 2.14 does not hold we can use this lemma and Theorem 2.13 to show (ii) and Theorem 2.13 suffices to show (iii). Parts (i) and (iv) would follow again by duality.

How do we do this in detail? Part (iii) follows from the fact that the cube and all its faces (which are again cubes) are quasi isotropic, that by projecting through a face of a cube all vertices stay on the circumsphere, and that the distance from the center of the cube to the center of any of its j -faces is $\sqrt{d - j}$.

To prove Part (ii) we remember the second statement after Lemma 2.9. First, we project T^{2d-1} onto $\sqrt{\frac{1}{2}}X^d$ by using the gsb

$$\sqrt{\frac{1}{2}} \begin{pmatrix} s_1 \\ -s_1 \end{pmatrix}, \dots, \sqrt{\frac{1}{2}} \begin{pmatrix} s_{d-1} \\ -s_{d-1} \end{pmatrix}, \begin{pmatrix} \mathbf{1}_{d-1} \\ -\mathbf{1}_{d-1} \end{pmatrix},$$

where s_1, \dots, s_{d-1} is any gsb for T^{d-1} . Now because for every even j , which is not excluded by both the theorem and the lemma, there exists a subset of size j of s_1, \dots, s_{d-1} , wlog s_1, \dots, s_j . But hence the sets

$$\sqrt{\frac{1}{2}} \begin{pmatrix} s_1 \\ -s_1 \end{pmatrix}, \dots, \sqrt{\frac{1}{2}} \begin{pmatrix} s_j \\ -s_j \end{pmatrix}$$

and

$$\sqrt{\frac{1}{2}} \begin{pmatrix} s_1 \\ -s_1 \end{pmatrix}, \dots, \sqrt{\frac{1}{2}} \begin{pmatrix} s_j \\ -s_j \end{pmatrix}, \begin{pmatrix} \mathbf{1}_{d-1} \\ -\mathbf{1}_{d-1} \end{pmatrix}$$

are gsb's in \mathbb{E}^{2d} and therefore there exists a projection of X^d onto any j' subspace such that it attains the lower bound $\sqrt{\frac{j'}{d}}$, except the case where $2j'$ or $2j' - 1$ does not pass the conditions of Theorem 2.13 or Lemma 2.14.

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